STATISTICAL ANALYSIS OF REPORTED TAG RECAPTURES FROM AN EXPLOITED POPULATION: MAXIMUM LIKELIHOOD ESTIMATION OF INSTANTANEOUS MORTALITY RATES

by

BU-370-M June, 1971

D. S. Robson

Abstract

One special case of the stochastic model for reported tag recaptures given in BU-369-M is a Poisson process with two competing risks of death. On the assumption or approximation of constant risks within each year and constant risk of natural mortality between years the survival rate becomes $S_j = \exp(-p_j - q)$ and $f_j = -\lambda_j p_j(\log S_j)/(1 - S_j)$. Only the relative values of the λ_j are identifiable from reported tag recaptures, and hence this transformation of parameters is one to one and results in no essential change in the statistical problem. The hypothesis that $\lambda_1 = \cdots = \lambda_{k-1}$ is testable, however, and this common value of λ is identifiable. Maximum likelihood estimation for this case leads to a chi-square test of this hypothesis against the alternative hypothesis of variable λ_j.

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1. Introduction

The model for reported tag recaptures used by Robson and Youngs (BU-369-M) may be specialized to the case where death is a Poisson process with two competing risks. If p_j is the average risk of death from fishing during year j (average instantaneous fishing mortality rate) and q_j is the average risk of death from natural causes then the annual survival rate S_j may be expressed as $-\log S_j = p_j + q_j$, and the annual rate of reported tag recaptures $f_j = \lambda_j u_j$ approximately satisfies the relation

$$\lambda_j p_j = - \frac{f_j \log S_j}{1 - S_j}$$

(if rates are constant within years then this relation is exact).

This transformation of parameters does not basically alter the statistical estimation problem, since the transformation is one to one, and the non-identifiability of λ_j is unaffected. If we assume a stable environmental situation, however, we might also assume that the risk q_j of natural mortality in the adult population likewise remains constant, $q_j = q$, and the hypothesis of constant reporting rates, H_o: $\lambda_j = \lambda$, then clearly becomes testable. In the case of an intensively exploited population of otherwise long lived individuals, the relation $\lambda_j \doteq f_j/(1 - S_j)$ will obtain and hence provide the basis for a crude check of the

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hypothesis $\lambda_j = \lambda$. In view of the high correlations among the estimates of f_j and S_j reported earlier, however, the need for statistical efficiency in this analysis is apparent, and we therefore turn to the maximum likelihood procedure.

2. Maximum likelihood estimators under H_o

The log likelihood function $L_o = \log P_{H_o}$ with respect to H_o becomes:

$$L_o = \sum_{i=1}^{k-1}\left[c_i \log\lambda + c_i \log u_i + (T_{i+1} - R_{i+1})\log S_i + (N_i - R_i)\log(1 - \rho_i)\right]$$
$$+ c_k \log f_k + (N_k - R_k)\log(1 - f_k)$$

where

$$S_i = e^{-(p_i+q)} \qquad\qquad u_i = \frac{p_i}{p_i + q}(1 - S_i)$$

$$\rho_i = \lambda\left[u_i + S_i u_{i+1} + S_i S_{i+1} u_{i+2} + \cdots + S_i S_{i+1} \cdots S_{k-2} u_{k-1}\right]$$
$$+ S_i S_{i+1} \cdots S_{k-1} f_k$$

$$\underset{\text{Def.}}{\equiv} \lambda\theta_i + S_i S_{i+1} \cdots S_{k-1} f_k \qquad \text{for } i = 1, \cdots, k - 1 \quad (\rho_k \equiv f_k).$$

The likelihood equations may then be expressed in terms of the partial derivatives

$$\frac{\partial u_i}{\partial q} = u_i' = \frac{p_i S_i - u_i}{p_i + q} \qquad\qquad \frac{\partial u_i}{\partial p_i} = u_i^* = \frac{u_i}{p_i} + u_i'$$

$$\frac{\partial\theta_i}{\partial q} = \theta_i' = u_i' + S_i u_{i+1}' + S_i S_{i+1} u_{i+2}' + \cdots + S_i S_{i+1} \cdots S_{k-2} u_{k-1}'$$
$$- S_i\left[u_{i+1} + 2S_{i+1} u_{i+2} + 3S_{i+1} S_{i+2} u_{i+3} + \cdots + (k - i - 1)S_{i+1} \cdots S_{k-2} u_{k-1}\right]$$

$$\frac{\partial \rho_i}{\partial q} = \rho_i' = \lambda\theta_i' - (k - i)S_i S_{i+1} \cdots S_{k-1} f_k$$

$$\frac{\partial \theta_i}{\partial p_j} = \theta_{ij}^* = (u_j^* - S_j\theta_{j+1})S_i S_{i+1} \cdots S_{j-1}$$

$$\frac{\partial \rho_i}{\partial p_j} = \rho_{ij}^* = \lambda\theta_{ij}^* - S_i S_{i+1} \cdots S_{k-1} f_k$$

giving

$$\frac{\partial L_o}{\partial \lambda} = \sum_{i=1}^{k-1}\left[\frac{c_i}{\lambda} - \frac{(N_i - R_i)\theta_i}{1 - \rho_i}\right] = 0$$

$$\frac{\partial L_o}{\partial q} = \sum_{i=1}^{k-1}\left[\frac{c_i u_i'}{u_i} - (T_{i+1} - R_{i+1}) - \frac{(N_i - R_i)\rho_i'}{1 - \rho_i}\right] = 0$$

$$\frac{\partial L_o}{\partial f_k} = \frac{c_k}{f_k} - \sum_{i=1}^{k}\frac{N_i - R_i}{1 - \rho_i} S_i S_{i+1} \cdots S_{k-1} = 0$$

$$\frac{\partial L_o}{\partial p_j} = \frac{c_j u_j^*}{u_j} - (T_{j+1} - R_{j+1}) - \sum_{i=1}^{j}\frac{(N_i - R_i)\rho_{ij}^*}{1 - \rho_i} = 0 \quad \text{for } j = 1, \cdots, k - 1.$$

These likelihood equations may be solved iteratively by conventional techniques. Letting Δ denote the (column vector of) partial derivatives of L_o,

$$\Delta' = \left(\frac{\partial L_o}{\partial \lambda}, \frac{\partial L_o}{\partial q}, \frac{\partial L_o}{\partial f_k}, \frac{\partial L_o}{\partial p_1}, \cdots, \frac{\partial L_o}{\partial p_{k-1}}\right)$$

and letting V^{-1} denote the information matrix

$$V^{-1} = \begin{bmatrix} -E\dfrac{\partial^2 L_o}{\partial\lambda^2} & -E\dfrac{\partial^2 L_o}{\partial\lambda\partial q} & -E\dfrac{\partial^2 L_o}{\partial\lambda\partial f_k} & \cdots & -E\dfrac{\partial^2 L_o}{\partial\lambda\partial p_{k-1}} \\ -E\dfrac{\partial^2 L_o}{\partial q\partial\lambda} & -E\dfrac{\partial^2 L_o}{\partial q^2} & -E\dfrac{\partial^2 L_o}{\partial q\partial f_k} & \cdots & -E\dfrac{\partial^2 L_o}{\partial q\partial p_{k-1}} \\ -E\dfrac{\partial^2 L_o}{\partial f_k\partial\lambda} & -E\dfrac{\partial^2 L_o}{\partial f_k\partial q} & -E\dfrac{\partial^2 L_o}{\partial f_k^2} & \cdots & -E\dfrac{\partial^2 L_o}{\partial f_k\partial p_{k-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ -E\dfrac{\partial^2 L_o}{\partial p_{k-1}\partial\lambda} & -E\dfrac{\partial^2 L_o}{\partial p_{k-1}\partial q} & -E\dfrac{\partial^2 L_o}{\partial p_{k-1}\partial f_k} & \cdots & -E\dfrac{\partial^2 L_o}{\partial p_{k-1}^2} \end{bmatrix}$$

then the estimates for the $(\nu + 1)^{th}$ iteration of the likelihood equations are obtained by adding $V_{(\nu)}\Delta_{(\nu)}$ to the estimates obtained from the ν^{th} iteration. Substituting the estimates from the ν^{th} iteration into V and Δ produces $V_{(\nu)}$ and $\Delta_{(\nu)}$, respectively.

The entries of V^{-1} may be expressed in terms of the second partial derivatives

$$\frac{\partial^2 u_i}{\partial q^2} = u_i'' = -\frac{p_i S_i + 2u_i'}{p_i + q}$$

$$\frac{\partial^2 u_i}{\partial p_i \partial q} = u_i^{*\prime} = u_i'' + \frac{u_i'}{p_i}$$

$$\frac{\partial^2 u_i}{\partial p_i^2} = u_i^{**} = u_i^{*\prime} + \frac{u_i^*}{p_i} - \frac{u_i}{p_i^2}$$

$$\theta_i'' = \frac{\partial^2\theta_i}{\partial q^2} = u_i'' + S_i u_{i+1}'' + S_i S_{i+1} u_{i+2}'' + \cdots + S_i S_{i+1} \cdots S_{k-2} u_{k-1}''$$

$$- 2S_i\left[u_{i+1}' + 2S_{i+1}u_{i+2}' + 3S_{i+1}S_{i+2}u_{i+3}' + \cdots + (k - 1 - i)S_{i+1} \cdots S_{k-2}u_{k-1}'\right]$$

$$+ S_i\left[u_{i+1} + 2^2S_{i+1}u_{i+2} + 3^2S_{i+1}S_{i+2}u_{i+3} + \cdots + (k - 1 - i)^2S_{i+1} \cdots S_{k-2}u_{k-1}\right]$$

$$\theta_{ij}^{*\prime} = \frac{\partial^2\theta_i}{\partial p_j \partial q} = \left[u_j^{*\prime} + S_j(\theta_{j+1} - \theta_{j+1}')\right] S_i S_{i+1} \cdots S_{j-1} - (j - i)\theta_{ij}^*$$

$$\theta_{ij}^{**} = \frac{\partial^2\theta_i}{\partial p_j^2} = \left[u_j^{**} + S_j\theta_{j+1}\right] S_i S_{i+1} \cdots S_{j-1}$$

$$\rho_i'' = \frac{\partial^2\rho_i}{\partial q^2} = \lambda\theta_i'' + (k - i)^2 S_i S_{i+1} \cdots S_{k-1} f_k$$

$$\rho_{ij}^{*\prime} = \frac{\partial^2\rho_i}{\partial p_j \partial q} = \lambda\theta_{ij}^{*\prime} + (k - i)S_i S_{i+1} \cdots S_{k-1} f_k$$

$$\rho_{ij}^{**} = \frac{\partial^2\rho_i}{\partial p_j^2} = \lambda\theta_{ij}^{**} + S_i S_{i+1} \cdots S_{k-1} f_k$$

Letting $\xi_i = N_1 S_1 S_2 \cdots S_{i-1} + N_2 S_2 \cdots S_{i-1} + \cdots + N_i$ we thus obtain for V^{-1}

$$-E\,\frac{\partial^2 L_o}{\partial\lambda^2} = \sum_{i=1}^{k-1}\left[\frac{\xi_i u_i}{\lambda} + \frac{N_i\theta_i^2}{1 - \rho_i}\right]$$

$$-E\,\frac{\partial^2 L_o}{\partial\lambda\partial q} = \sum_{i=1}^{k-1} \frac{N_i}{1 - \rho_i}\left[(1 - \rho_i)\theta_i' + \theta_i\rho_i'\right]$$

$$-E\,\frac{\partial^2 L_o}{\partial\lambda\partial f_k} = \sum_{i=1}^{k-1} \frac{N_i\theta_i S_i S_{i+1} \cdots S_{k-1}}{(1 - \rho_i)}$$

$$-E\,\frac{\partial^2 L_o}{\partial\lambda\,\partial p_j} = \sum_{i=1}^{j} \frac{N_i}{1-\rho_i}\left[(1-\rho_i)\theta^*_{ij} + \theta_i\rho^*_{ij}\right]$$

$$-E\,\frac{\partial^2 L_o}{\partial q^2} = \sum_{i=1}^{k-1}\left\{\frac{\lambda\xi_i}{u_i}\left[(u_i')^2 - u_i u_i''\right] + \frac{N_i}{1-\rho_i}\left[\rho_i''(1-\rho_i) + (\rho_i')^2\right]\right\}$$

$$-E\,\frac{\partial^2 L_o}{\partial q\,\partial f_k} = \sum_{i=1}^{k-1}\frac{N_i S_i S_{i+1}\cdots S_{k-1}}{1-\rho_i}\left[\rho_i' - (k-i)(1-\rho_i)\right]$$

$$-E\,\frac{\partial^2 L_o}{\partial q\,\partial p_j} = \frac{\lambda\xi_j}{u_j}\left[u^*_j u_j' - u_j u^{*\prime}_j\right] + \sum_{i=1}^{j}\frac{N_i}{1-\rho_i}\left[\rho^{*\prime}_{ij}(1-\rho_i) + \rho_i'\rho^*_{ij}\right]$$

$$-E\,\frac{\partial^2 L_o}{\partial f_k^2} = \frac{\xi_k}{f_k} + \sum_{i=1}^{k}\frac{N_i(S_i S_{i+1}\cdots S_{k-1})^2}{(1-\rho_i)}$$

$$-E\,\frac{\partial^2 L_o}{\partial f_k\,\partial p_j} = \sum_{i=1}^{j}\frac{N_i S_i S_{i+1}\cdots S_{k-1}}{1-\rho_i}\left[\rho_i + \rho^*_{ij} - 1\right]$$

$$-E\,\frac{\partial^2 L_o}{\partial p_h\,\partial p_j} = \sum_{i=1}^{h}\frac{N_i\rho^*_{ij}}{1-\rho_i}(\rho_i + \rho^*_{ih} - 1)$$

for $h < j$, and

$$-E\,\frac{\partial^2 L_o}{\partial p_j^2} = \frac{\lambda\xi_j}{u_j}\left[(u^*_j)^2 - u_j u^{**}_j\right] + \sum_{i=1}^{j}\frac{N_i}{1-\rho_i}\left[\rho^{**}_{ij}(1-\rho_i) + (\rho^*_{ij})^2\right].$$

The matrix V used in the terminal iteration is then the estimated covariance matrix of the maximum likelihood estimators. Convenient initial values for starting the iteration are $f_k = \hat{f}_k$ and

$$q = 0 \qquad p_j = -\log \hat{S}_j \qquad \lambda = \frac{1}{k-1} \sum_{i=1}^{k-1} \frac{\hat{f}_i}{1 - \hat{S}_i}$$

where $\hat{f}_i$ and $\hat{S}_i$ are the estimators given in Section 3 of BU-369-M.

3. A chi-square test of the hypothesis H_o

The hypothesis H_o: $\{q_j = q,\ \lambda_j = \lambda \text{ for } j = 1, \cdots, k-1\}$ may be tested against the alternative H_1: $\{q_j = q \text{ for } j = 1, \cdots, k-1\}$ by the likelihood ratio test or, equivalently (asymptotically), by a chi-square "reduction in goodness of fit test". The hypothesis H_1 is isomorphic to the hypothesis of Section 2 in BU-369-M; i.e., the parameters of H_1 are a one to one transformation of the parameters $(f_1, \cdots, f_k, S_1, \cdots, S_{k-1})$ of BU-369-M. Only the relative values of $\lambda_1, \cdots, \lambda_{k-1}$ are identifiable with respect to H_1; hence, assigning an arbitrary value to one of the λ_j's, say $\lambda_1 = 1$, we have the transformation:

$$p_1 = -\frac{f_1 \log S_1}{1 - S_1}, \qquad q = -\log S_1 - p_1,$$

$$p_j = -\log S_j - q, \qquad \lambda_j = -\frac{f_j \log S_j}{p_j(1 - S_j)}$$

for $j = 1, \cdots, k-1$, and

$$\lambda_k u_k = f_k .$$

The chi-square test of Section 4 thus tests goodness of fit to hypothesis H_1 above. An alternative and asymptotically equivalent test of goodness of fit to H_1 could be obtained by the more tedious but also more heuristic method of substituting the parameter estimates $(\hat{f}_1, \cdots, \hat{f}_k, \hat{S}_1, \cdots, \hat{S}_{k-1})$ into the formulas for $E_{H_1}(R_{ij})$

and computing

$$\chi^2_{(k-1)(k-2)/2} = \sum_{i<j}^{k} \frac{[R_{ij} - \hat{E}_{H_1}(R_{ij})]^2}{\hat{E}_{H_1}(R_{ij})} .$$

Similarly, substituting the maximum likelihood estimates $(\hat{\lambda}, \hat{q}, \hat{f}_k, \hat{p}_1, \cdots, \hat{p}_{k-1})$ obtained by iteration into the formulas $E_{H_o}(R_{ij})$ and computing

$$\chi^2_{(k^2-k-4)/2} = \sum_{i<j}^{k} \frac{[R_{ij} - \hat{E}_{H_o}(R_{ij})]^2}{\hat{E}_{H_o}(R_{ij})}$$

provides a chi-square test of goodness of fit to H_o. The difference

$$\chi^2_{k-3} = \chi^2_{(k^2-k-4)/2} - \chi^2_{(k-1)(k-2)/2}$$

measuring the reduction in goodness of fit between H_1 and H_o then provides a test of H_o against H_1.